1. For $A = \begin{pmatrix} 1 & i & 0 \\ -i & 1 & i \\ 0 & -i & 1 \end{pmatrix} \in M_{3\times 3}(\mathbb{C})$, find a unitary matrix P and a diagonal matrix D such that $P^*AP = D$.

$$f_A(t) = \det(A - t\cdot I_3) = \begin{vmatrix} 1-t & i & 0 \\ -i & 1-t & i \\ 0 & -i & 1-t \end{vmatrix}$$

$$= (1-t)\left((1-t)^2 - 1\right) - i\cdot(-i)(1-t)$$

$$= (1-t)\left((1-t)^2 - 2\right) = (1-t)(1-t+\sqrt{2})(1-t-\sqrt{2})$$

$\lambda_1 = 1 \qquad \lambda_2 = 1+\sqrt{2} \qquad \lambda_3 = 1-\sqrt{2}$

- $A - \lambda_1 I_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad v_1 = u_1 / \|u_1\| = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot / \sqrt{2}$

- $A - \lambda_2 \cdot I_3 = \begin{pmatrix} -\sqrt{2} & i & 0 \\ -i & -\sqrt{2} & i \\ 0 & -i & -\sqrt{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & -\sqrt{2} & 2i \\ 0 & 0 & 0 \end{pmatrix}$

$u_2 = \begin{pmatrix} -1 \\ \sqrt{2}i \\ 1 \end{pmatrix} \qquad v_2 = u_2 / \|u_2\| = \begin{pmatrix} -1 \\ \sqrt{2}i \\ 1 \end{pmatrix} / 2$

- $A - \lambda_3 I_3 = \begin{pmatrix} \sqrt{2} & i & 0 \\ -i & \sqrt{2} & i \\ 0 & -i & \sqrt{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 2i \\ 0 & 0 & 0 \end{pmatrix}$

$u_3 = \begin{pmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{pmatrix} \qquad v_3 = u_3 / \|u_3\| = \begin{pmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{pmatrix} / 2$

$P = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ is unitary. $D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$

$A\cdot P = P\cdot D$ i.e. $P^* A P = D$

2. Sec. 6.5 Q9

9. Let U be a linear operator on a finite-dimensional inner product space V. If $\|\mathsf{U}(x)\| = \|x\|$ for all x in some orthonormal basis for V, must U be unitary? Justify your answer with a proof or a counterexample.

Consider $V = \mathbb{R}^2$. $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $U = L_A$

$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is o.n. basis for V.

Then $\|U(x)\| = \|x\|$ for $x \in \beta$.

But $\left\| U\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right\| = 2 \neq \sqrt{2} = \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|$

So U is not unitary

3. Sec. 6.5 Q19

19. Let W be a finite-dimensional subspace of an inner product space V. By Theorem 6.7 (p. 352) and the exercises of Section 1.3, $\mathsf{V} = \mathsf{W} \oplus \mathsf{W}^{\perp}$. Define $\mathsf{U}\colon \mathsf{V} \to \mathsf{V}$ by $\mathsf{U}(v_1 + v_2) = v_1 - v_2$, where $v_1 \in \mathsf{W}$ and $v_2 \in \mathsf{W}^{\perp}$. Prove that U is a self-adjoint unitary operator.

$\forall x, y \in V = W \oplus W^{\perp}$. $\exists!\ x_1, y_1 \in W$ and $!\ x_2, y_2 \in W^{\perp}$

such that $x = x_1 + x_2$, $y_1 + y_2$.

Then $U(x) = U(x_1 + x_2) = x_1 - x_2$

$U(y) = U(y_1 + y_2) = y_1 - y_2$

$$\begin{cases} \langle U(x), y \rangle = \langle x_1 - x_2, y_1 + y_2 \rangle = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle \\ \langle x, U(y) \rangle = \langle x_1 + x_2, y_1 - y_2 \rangle = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle \end{cases}$$

Thus $\langle U(x), y \rangle = \langle x, U(y) \rangle$ i.e. $U = U^*$

$$\begin{cases} \|U(x)\|^2 = \langle U(x), U(x) \rangle = \langle x_1 - x_2, x_1 - x_2 \rangle = \langle x_1, x_1 \rangle + \langle x_2, x_2 \rangle \\ \|x\|^2 = \langle x, x \rangle = \langle x_1 + x_2, x_1 + x_2 \rangle = \langle x_1, x_1 \rangle + \langle x_2, x_2 \rangle \end{cases}$$

Thus $\|U(x)\| = \|x\|$ $\forall x \in V$.

4. Sec. 6.6 Q4

4. Let W be a finite-dimensional subspace of an inner product space V. Show that if T is the orthogonal projection of V on W, then I − T is the orthogonal projection of V on $W^{\perp}$.

$W \subset V$, W is finite-dim.

If T is orth proj of V on W.

Then $R(T) = W$, $R(T)^{\perp} = N(T)$, $R(T) = N(T)^{\perp}$

Let $U = I - T$

① Claim: $R(U) = W^{\perp}$

- $\forall v \in V$ $\quad U(v) = v - T(v)$

  $T(U(v)) = T(v) - T^2(v) = 0$

  So $U(v) \in N(T) = W^{\perp}$ i.e. $R(U) \subset W^{\perp}$

- $\forall x \in W^{\perp} = N(T)$, $\exists x \in V$. s.t $U(x) = x - T(x) = x$

  So $x \in R(U)$ i.e. $W^{\perp} \subset R(U)$

② Claim: $N(U) = W$

- $\forall x \in W$. $\quad U(x) = x - T(x) = x - x = 0$.

  So $x \in N(U)$ i.e. $W \subset N(U)$

- $\forall x \in N(U)$. $\quad U(x) = 0 \Rightarrow x = T(x) \in R(T) = W$

  So $N(U) \subset W$

Therefore $\quad R(U) = W^{\perp} = N(U)^{\perp}$

$$N(U) = W = (W^{\perp})^{\perp} = R(U)^{\perp}$$

since W is finite-dim

5. Sec. 6.6 Q7(c)(d)(f)

7. Let T be a normal operator on a finite-dimensional complex inner product space V. Use the spectral decomposition $\lambda_1\mathsf{T}_1 + \lambda_2\mathsf{T}_2 + \cdots + \lambda_k\mathsf{T}_k$ of T to prove the following results.

(a) If g is a polynomial, then

$$g(\mathsf{T}) = \sum_{i=1}^{k} g(\lambda_i)\mathsf{T}_i.$$

(b) If $\mathsf{T}^n = \mathsf{T}_0$ for some n, then $\mathsf{T} = \mathsf{T}_0$.

(c) Let U be a linear operator on V. Then U commutes with T if and only if U commutes with each T_i.

(d) There exists a normal operator U on V such that $\mathsf{U}^2 = \mathsf{T}$.

(e) T is invertible if and only if $\lambda_i \neq 0$ for $1 \leq i \leq k$.

(f) T is a projection if and only if every eigenvalue of T is 1 or 0.

(g) $\mathsf{T} = -\mathsf{T}^*$ if and only if every λ_i is an imaginary number.

(c) Suppose U commutes with each T_i. Then we have

$$\begin{aligned} UT &= U\left(\sum_{i=1}^{k} \lambda_i T_i\right) \\ &= \sum_{i=1}^{k} \lambda_i U T_i \\ &= \sum_{i=1}^{k} \lambda_i T_i U \\ &= \left(\sum_{i=1}^{k} \lambda_i T_i\right) U = TU \end{aligned}$$

Conversely, suppose U commutes with T. Note that for each T_i, there exists some polynomial g_i such that $g_i(T) = T_i$. Then we have

$$UT_i = Ug_i(T) = g_i(T)U = T_iU.$$

(d) Note that $T_iT_j = \delta_{ij}T_j$ and $T = \sum_{i=1}^{k} \lambda_i T_i$. Let

$$U = \sum_{i=1}^{k} \lambda_i^{\frac{1}{2}} T_i.$$

Then one can easily check that $U^2 = T$. Since T_i are self-adjoint, that is T_i is normal, thus U is normal, too.

(e) Note that V is finite-dimensional. So T is invertible if and only if $N(T) = 0$. But this means 0 is not an eigenvalue of T.

(f) Suppose T is a projection of V on W along $W^\perp$. Let λ be eigenvalue and $v \in V$ be the corresponding eigenvector. Then there is some $w \in W$ and $y \in W^\perp$ such that $v = w + y$. So, we have

$$\begin{aligned} w = T(w+y) &= \lambda(w+y) \\ (1-\lambda)w &= \lambda y. \end{aligned}$$

This means that λ can only be 1 or 0.